CONTINUOUS SELECTIONS WITH RESPECT TO EXTENSION DIMENSION

A. V. KARASEV

Abstract. Let L be finite CW complex. By $[L]$ we denote extension type of L. The following generalization of Michael's selection theorem is proved:

Theorem. *Consider $[L] \le [S^n]$. Let F be a lower semi-continuous map between polish space X and metrizable compactum Y, such that $\operatorname{ed}(X) \le [L]$, F is equi-$LC^{[L]}$ collection and $F(x) \in \mathrm{AE}([L])$ for any $x \in X$. Let A be a closed subset of X such that there exists a continuous selection $f : A \to Y$ of $F|_A$. Then F admits a continuous selection $\bar{f}$ which extends f.*

1. Introduction

The following Michael's selection theorem is well-known (see [1] for details):

Theorem 1.1. *Let X be a paracompact space, $A \subseteq X$ a closed subspace of X with $\dim_X(X - A) \le n + 1$, Y a complete metric space, F an equi-$LC^n \bigcap C^n$ collection and $F : X \to Y$ lower semi-continuous map. Then every selection for $F|_A$ can be extended to a selection for F.*

Theorem 1.1 deals with usual Lebesgue dimension and concerns the notion of absoulute extensor in dimension n. The purpose of the present paper is to obtain a natural generalization of Michael's theorem in the case of extension dimension.

2. Preliminaries

In this part we introduce notions of *extension types of complexes, extension dimension, absolute extensors modulo a complex, $[L]$-homotopy* and *equi-$LC^{[L]}$ collections*. All spaces are polish, all complexes are countable finitely-dominated CW complexes. For more details related to extension dimension see [2].

1991 *Mathematics Subject Classification.* Primary: 54C65; Secondary: 54C20, 54F45.

Key words and phrases. Continuous selections, extension dimension, dimension.

For spaces X and L, the notation $L \in \mathrm{AE}(X)$ means, that every map $f : A \to L$, defined on a closed subspace A of X, admits an extension $\bar{f}$ over X.

Let L and K be complexes. We say (see [2]) that $L \leq K$ if for each space X from $L \in \mathrm{AE}(X)$ follows $K \in \mathrm{AE}(X)$. Equivalence classes of complexes with respect to this relation are called *extension types*. By $[L]$ we denote extension type of L.

Definition 2.1. ([2]). The extension dimension of a space X is extension type $\mathrm{ed}(X)$ such that $\mathrm{ed}(X) = \min\{[L] : L \in \mathrm{AE}(X)\}$.

Observe, that if $[L] \leq [S^n]$ and $\mathrm{ed}(X) \leq [L]$, then $\dim X \leq n$.

Definition 2.2. ([2]). We say that a space X is *an absolute extensor modulo* L (shortly X is $\mathrm{AE}([L])$) and write $X \in \mathrm{AE}([L])$ if $X \in \mathrm{AE}(Y)$ for each space Y with $\mathrm{ed}(X) \leq [L]$.

We will widely use the following proposition:

Proposition 2.1. ([2]). *Let X be a polish space such that* $\mathrm{ed}(X) \leq [L]$ *and* $Y \in \mathrm{AE}([L])$. *Then* $L \in \mathrm{AE}(X')$ *for any* $X' \subseteq X$.

Follow [2] give definition of $[L]$-*homotopy* and u-$[L]$-homotopy:

Definition 2.3. Two maps $f_0, f_1 : X \to Y$ are said to be $[L]$-homotopic (notation: $f_0 \overset{[L]}{\simeq} f_1$) if for any map $h : Z \to X \times [0,1]$, where Z is a space with $\mathrm{ed}(Z) \leq [L]$, the composition $(f_0 \oplus f_1)h|_{h^{-1}(X\times\{0,1\})} : h^{-1}(X \times \{0,1\}) \to Y$ admits an extension $H : Z \to Y$. If, in addition, we are given $\mathcal{U} \in \mathrm{cov}(Y)$ and H can be choosen so that the collection $\{H(h^{-1}(\{x\} \times [0,1])) : x \in X\}$ refines $\mathcal{U}$, we say, that f_0 and f_1 are $\mathcal{U}$-$[L]$-homotopic, and write $f_0 \overset{\mathcal{U}-[L]}{\simeq} f_1$.

It is clear, that if f_0, f_1 are $\mathcal{U}$-$[L]$-homotopic for some $[L]$ then these maps are $\mathcal{U}$-close.

Let us observe (see [2]) that $\mathrm{AE}([L])$-spaces have the following important property:

Proposition 2.2. *Let Y be a Polish* $\mathrm{AE}([L])$*-space. Then for each* $\mathcal{U} \in \mathrm{cov}(Y)$ *there exists* $\mathcal{V} \in \mathrm{cov}(Y)$ *refining* $\mathcal{U}$*, such that for any space X with* $\mathrm{ed}(X) \leq [L]$*, any closed subspace $A \subseteq X$ and any two $\mathcal{V}$-close maps $f, g : A \to Y$ from existance of extension $\bar{f}$ of f over X follows existance $\bar{g}$ which is $\mathcal{U}$-close to $\bar{f}$ and extends g over X.*

Corollary 2.3. *Let Y be a copmact* $\mathrm{AE}([L])$*-space.*
Then for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for any space X with $\mathrm{ed}(X) \leq [L]$*, any closed subspace $A \subseteq X$ and any two δ-close*

maps $f, g : A \to Y$ from existance of extension $\bar{f}$ of f over X follows existance $\bar{g}$ which is ε-close to $\bar{f}$ and extends g over X.

From just mentioned fact one can easely obtain the following:

Proposition 2.4. *Let Y be a metrizable* $\mathrm{AE}([L])$ *compactum. Then for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for any space X with* $\mathrm{ed}(X) \leq [L]$*, any closed subspace $A \subseteq X$ and any map $f : A \to Y$ such that* $\mathrm{diam}(f(A)) \leq \delta$ *there exists $\bar{f} : X \to Y$ extending f such that* $\mathrm{diam}(\bar{f}(X) \leq \varepsilon$.

The last proposition allows us to introduce in the natural way the notion of *equi-$LC^{[L]}$ collection*.

Definition 2.4. Collection $\mathfrak{F} = \{F_\alpha : \alpha \in \mathfrak{A}\}$ of closed sabsets of compact space Y is said to be equi-$LC^{[L]}$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for each $\alpha \in \mathfrak{A}$, each Polish space Z with $\mathrm{ed}(Z) \leq [L]$, each closed subset $A \subseteq Z$ and a map $f : A \to F_\alpha$ such that $\mathrm{diam}(f(A)) < \delta$ there exists an extension $\bar{f} : Z \to F_\alpha$ of f over Z such that $\mathrm{diam}\, f(Z) < \varepsilon$.

3. Selection theorem

Let us recall that a many-valued map $F : X \to Y$ is said to be lower semi-continuous (shortly l.s.c.) if $F(x)$ is closed subset of Y for any $x \in X$ and for any open $U \subseteq Y$ the set $F^{-1}(U) = \{x \in X : F(x) \bigcap U \neq \varnothing\}$ is open in X.

We are ready now to formulate our main result.

Theorem 3.1. *Let Y be a metrizable compactum, X a Polish space with* $\mathrm{ed}(X) \leq [L]$ *and $F : X \to Y$ a l.s.c. map such that collection $\mathfrak{F} = \{F(x) : x \in X\}$ is equi-$LC^{[L]}$ and $F(x) \in$* $\mathrm{AE}([L])$ *for each $x \in X$. Let $A \subseteq X$ be a closed subset. Then any selection $f : A \to Y$ of $F|_A$ can be extended to selection $\bar{f} : X \to Y$.*

Remark 3.1. Important difference between the further proof of Theorem 3.1 and consideration of [1] consists in the fact that we cannot apply technique of [1] involving maps into nerves of covering. We have to directly extend maps over open subspaces of X and hence we need to use Proposition 2.1. Therefore proof presented in this text cannot be directly generalized on the case when X is paracompact space.

Further, we have no characterization of absolute extensors modulo $[L]$ in terms of maps of spheres. It makes Proposition 2.4 and in turn compactness of Y essential for our consideration.

It is also necessary to point out, that in [2] $[L]$-dimensional analogies of n-dimensional spheres were introduced, namely, a compact spaces

$S^n_{[L]}$, which are ANE($[L]$) and admit $[L]$-invertable and approximately $[L]$-soft mappings onto n-dimensional sphere (see [2] for necessary definition). Additionaly, these spaces are proved to be $[L]$-universal for compact spaces. This fact, it would seem, allows to introduce the notion of equi-$LC^{[L]}$ families using characterization in terms of mappings of $S^n_{[L]}$ which were closer to original definition in [1], and generalize the theorem on the case of non-compact Y.

Unfortunately, as it already has been mentioned above, our proof involve extensions of maps over open subspaces of X which are non-compact. Therefore we cannot use universality of $S^n_{[L]}$.

Simillar [1], we accomplish the proof of this theorem consequently reducing it to other assertion. Using arguments of [1], one can easely observe, that Theorem 3.1 is equivalent to the following

Theorem 3.2. *Let $Y = Q$ be Hilbert cube, X a Polish space with $\mathrm{ed}(X) \leq [L]$ and $F : X \to Q$ an l.s.c. map such that collection $\mathcal{F} = \{F(x) : x \in X\}$ is equi-$LC^{[L]}$ and $F(x) \in \mathrm{AE}([L])$ for each $x \in X$. Then F admits selection $f : X \to Y$.*

Let $B \subseteq Y$. By $O_\varepsilon(B)$ we denote ε-nighbourhood of B in Y.

Finally, let us reduce Theorem 3.2 to the following lemma:

Main Lemma. *Let X, Y and F be the same as in Theorem 3.2. Then*

- **a**. *For any $\mu > 0$ there exists $g : X \to Y$, which is μ-close to F.*
- **b**. *For any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ with the following property: for each $f : X \to Y$ such that f is δ-close to F and for each $\mu > 0$ there exists $g : X \to Y$ such that g is ε-close to f and μ-close to F.*

Let us prove that

Proposition 3.3. *Main Lemma implies Theorem 3.2.*

Proof. Consider a sequence $\varepsilon_n = \frac{1}{2^n}$, $n \geq 0$. Using Main Lemma construct corresponding sequences of $\{\delta_n < \varepsilon_n\}$, where $\delta_n = \delta_n(\varepsilon_n)$ and $\{f_n\}$ such that f_n is ε_n-close to f_{n-1}, $n \geq 1$ and δ_n close to F for every n. Then f_n is uniformly Cauchy. Since Y is metrizable compactum (actually we assume that Y is Hilbert cube), there exists continuous $f = \lim\limits_{n\to\infty} f_n$. Obviously, f is selection of F. □

4. Covers of special type

Let us introduce notations and definitions which are necessary to prove Main Lemma.

Since this point and up to the end of the text we assume that L is a complex such that $[L] \leq [S^n]$, X is a Polish space with $\mathrm{ed}(X) \leq [L]$ (and therefore with $\dim Xf \leq n$), Y is Hilbert cube (actually we need only the property $Y \in \mathrm{AE}$ and compactness of Y) and $F : X \to Y$ as in formulation of Main Lemma.

Definition 4.1. Let $\mathcal{U} \in \mathrm{cov}(X)$. Then $\mathcal{V} \in \mathrm{cov}(X)$ is said to be *a canonical refinment* for $\mathcal{U}$ if $\mathcal{V}$ satisfies the following conditions:

1. $\mathcal{V}$ is star-refinment of $\mathcal{U}$.
2. $\mathcal{V}$ is star-finite.
3. Order of $\mathcal{V}$ is $\leq n+1$.
4. $\mathcal{V}$ is irreducible, i.e. for any $V \in \mathcal{V}$ collection $\mathcal{V} \backslash \{V\}$ is not a cover of X.

Observe, that canonical refinment exists for any $\mathcal{U} \in \mathrm{cov}(X)$ (see [3] for details).

For any $\varepsilon > 0$ let $U_x = \{x' \in X : F(x) \subseteq O_\varepsilon(F(x'))\}$. Since F is l.s.c., U_x is open for any $x \in X$ [1].

Definition 4.2. Let $\mathcal{U} \in \mathrm{cov}(X)$. Then we say that $\mathcal{V} \in \mathrm{cov}(X)$ is *a canonical refinment for $\mathcal{U}$ with respect to F and ε* (notation: $\mathcal{V} = \mathcal{V}(U, F, \varepsilon)$, if it satisfies the following conditionsf:

1. $\mathcal{V}$ is star-refinment of $\mathcal{U}$.
2. For any $V \in \mathcal{V}$ there exists $x(V) \in V$ such that $V \subseteq U_{x(V)}$.

It is easy to see, that for any $\mathcal{U} \in \mathrm{cov}(X)$ and $\varepsilon > 0$ there exists a canonical refinment with respect to F and ε.

Let $\mathcal{U} = \{U_\alpha : \alpha \in \mathfrak{A}\}$ be a star-finite irreducible cover of X having oder $\leq n+1$.

Let $F_k = \{x \in X : \mathrm{ord}_{\mathcal{U}}\, x \leq k\}$, $k = 1 \ldots n+1$. Observe, that

F1. F_k is closed for any k.
F2. $F_k \subseteq F_{k+1}$ for any $k = 1 \ldots n$.
F3. $X = F_{n+1}$.

Further, for each $k = 1 \ldots n+1$ and $\mathfrak{B} \subseteq \mathfrak{A}$ such that $|\mathfrak{B}| = k$ let $G_k^{\mathfrak{B}} = F_k \bigcap (\bigcap \{U_\alpha : \alpha \in \mathfrak{B}\})$.

Notice that generally speaking, $G_k^{\mathfrak{B}}$ may be empty or non-closed.

Obviously, family $\mathcal{G}_k = \{G_k^{\mathfrak{B}} : |\mathfrak{B}| = k\}$ has the following properties:

G1. $\{G_1^{\{\alpha\}} : \alpha \in \mathfrak{A}\}$ are closed, pairwise disjoint and non-empty subsets of X.
G2. $\mathcal{G}_k$ is discrete in itself.
G3. $F_{k+1} = (\bigcup \mathcal{G}_{k+1}) \bigcup F_k$
G4. For each non-empty $G_{k+1}^{\mathfrak{B}}$, $\bigcup \{G_{|\mathfrak{B}'|}^{\mathfrak{B}'} : \mathfrak{B}' \subsetneq \mathfrak{B}\} \supseteq \overline{G_{k+1}^{\mathfrak{B}}} \bigcap F_k \neq \varnothing$

We will use these consideration as well as introduced notations in all the remaining text.

5. Technical lemmas

The following two lemmas we need to complete the proof are analogies of lemmas containing in Appendix of [1].

Lemma 5.1. *Let B be closed subset of Y, such that $B \in \mathrm{AE}([L])$. Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for every Polish space X with $\mathrm{ed}(X) \leq [L]$ and for each $f : X \to O_\delta(B)$ there exists $g : X \to B$ such that g is ε-close to f.*

Proof. Construct a sequence $\{\delta_k\}_{k=1}^{n+1}$ such that:

1δ. $\delta_{n+1} = \varepsilon$.
2δ. $\delta_k < \delta_{k+1}$.
3δ. Pair $(\frac{\delta_{k+1}}{3}, \delta_k)$ satisfies condition of Proposition 2.4.

Let $\delta = \frac{\delta_1}{6}$. Check that pair (ε, δ) satisfies requirments of lemma. Consider $f : X \to O_\delta(B)$.

Let $\mathcal{O} = \{O_y = O_\delta(y) : y \in B\}$ (since B is compact, we may choose finite refinment of $\mathcal{O}$, but it is not essential for further consideration). Let $V_y = f^{-1}(O_y)$ and $\mathcal{V} = \{V_y : y \in B\}$. Consider $\mathcal{U} \in \mathrm{cov}(X)$, which is canonical refinement of $\mathcal{V}$ in the sense of Definition 4.1. Let $\mathcal{U} = \{U_\alpha : \alpha \in \mathfrak{A}\}$. Using property 1 of canonical refinement, for each $\alpha \in \mathfrak{A}$ find y_α such that $\mathrm{St}_{\mathcal{U}} U_\alpha \subseteq V_{y_\alpha}$. Notice that generally speaking, y_α and y_β may coincide for $\alpha \neq \beta$. Finally, consider related to $\mathcal{U}$ sets F_k and $G_k^{\mathfrak{B}}$, introduced in Section 4.

We are ready now to construct map g.

Using induction by $k = 1 \dots n+1$ construct a sequence of map $\{g_k\}_{k=1}^{n+1}$ such that:

1g. $g_k : F_k \to B$.
2g. $g_{k+1}|_{F_k} = g_k$.
3g. $f|_{F_k}$ is ε-close to g_k.
4g. $g_k(F_k \bigcap U_\alpha) \subseteq O_{\delta_k/3}(y_\alpha)$ for each $\alpha \in \mathfrak{A}$.

For $k = 1$ define g_1 letting $g_1|_{G_1^{\{\alpha\}}} \equiv y(\alpha)$. Observe, that g_1 is defined correctly and continuous on F_1 (see properties G1–G4 on the page 5). By our choice of $\{y_\alpha\}$, g_1 satisfies conditions 1g–4g.

Assuming that g_k has been already constructed, let us construct g_{k+1}.

To accomplish this it is enough (see G3 on the page 5) to define g_{k+1} on each non-emty $G_{k+1}^{\mathfrak{B}}$ for each $\mathfrak{B} \subseteq \mathfrak{A}$ such that $|\mathfrak{B}| = k+1$.

Fix $\mathfrak{B}$ such that $|\mathfrak{B}| = k+1$. Consider $Z = \bigcup\limits_{\alpha \in \mathfrak{B}} U_\alpha$.

Let $Z' = Z \bigcap F_k$. Obviously, Z' is closed subset of Z. Since $U_\alpha \bigcap U_\beta \neq \varnothing$ for $\alpha, \beta \in \mathfrak{B}$ (recall that we consider $G^{\mathfrak{B}}_{k+1} \neq \varnothing$) by our choice of y_α we have $\mathrm{dist}(y_\alpha, y_\beta) < \delta$. Therefore, by property 4g we conclude that $\mathrm{diam}\, Z' < 2\delta + 2(\delta_k/3) < \delta_k$. Hence by our choice of $\{\delta_k\}$, map g_k has an extension $\bar{g}_k : Z \to B$ such that $\mathrm{diam}\, \bar{g}_k(Z) < \delta_{k+1}/3$. Let $g_{k+1}|_{G^{\mathfrak{B}}_{k+1}} \equiv \bar{g}_k$. Observe, that g_{k+1} is continuous (see property G2 on the page 5). Check that g_{k+1} satisfies conditions 1g–4g. Indeed, 1g and 2g are met by construction. Further, we have $\mathrm{diam}\, g_{k+1}(G^{\mathfrak{B}}_{k+1}) < \delta_{k+1}/3$, therefore, since $y_\alpha \in g_{k+1}(U_\alpha)$ for every $\alpha \in \mathfrak{A}$, condition 4g is also met. Finally, our choice of δ and $\{\delta_k\}$ coupled with property 3g for g_k and just checked property 4g for g_{k+1} (as well as the choice of y_α) yields the property 3g for g_{k+1}.

Since $F_{n+1} = X$ (see property F3 on the page 5) we complete the proof letting $g \equiv g_{n+1}$. □

Lemma 5.2. *Let B be a closed subset of* AE*-compactum Y, such that $B \in \mathrm{AE}([L])$. Then*

a. *For any $\mu > 0$ there exists $\nu = \nu(\mu) > 0$ such that for any X with $\mathrm{ed}(X) \leq [L]$, any $A \subseteq X$ closed in X and any map $f : A \to O_\nu(B)$ there exists $\bar{f} : X \to O_\mu(B)$ extending f.*

b. *For any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for any $\mu > 0$ there exists $\nu = \nu(\varepsilon, \mu)$ with the following property:*

for every X with $\mathrm{ed}(X) \leq [L]$, any $A \subseteq X$ closed in X and any $f : A \to O_\nu(B)$ such that $\mathrm{diam}\, f(A) < \delta$ there exists $\bar{f} : A \to O_\mu(B)$ such that $\mathrm{diam}\, f(X) < \varepsilon$.

Proof. **a.** Since Y is AE-compactum, for $\mu > 0$ pick $\xi = \xi(\mu)$ such that pair (μ, ξ) satisfies conditions of Corollary 2.3 for space Y. Further, for $\xi > 0$ choose ν such that pair (ξ, ν) meets conditions of Lemma 5.1. Check that pair (μ, ν) satisfies condition **a**.

Consider $f : A \to O_\nu(B)$. By our choice of ν there exists $g : A \to B$ such that $\mathrm{dist}(f, g) < \xi$. Since $B \in \mathrm{AE}(X)$ there exists extension $\bar{g} : X \to B$. Therefore, by our choice of ξ, there exists $\bar{f} : X \to Y$ such that $\mathrm{dist}(\bar{f}, \bar{g}) < \mu$.

The last fact implies that $f(X) \subseteq O_\mu(B)$.

b. Consider $\varepsilon' = \varepsilon/4$. For ε' pick $\delta' = \delta(\varepsilon') > 0$ such that pair (ε', δ') satisfies conditions of Proposition 2.4 for space B. Let $\delta = \delta'/4$. Observe, that $\delta = \delta(\varepsilon)$.

Further, let $\lambda = \min(\mu, \frac{\varepsilon}{3})$. For λ find $\xi > 0$ as in Corollary 2.3, applied to space Y. We may assume that $\xi < \delta$. For $\xi > 0$ pick $\nu = \nu(\xi) > 0$ as in Lemma 5.1.

Check, that δ and ν satisfy our requirments.

Consider $f : A \to O_\nu(B)$ such that $\operatorname{diam} f(A) < \delta$. By the choice of ν there exists $g : A \to B$, which is ξ-close to f and hence δ-close to f. This fact implies, that $\operatorname{diam} g(A) < 3\delta = \delta'$, which, in turn, implies by the choice of δ', that g has an extension $\bar{g} : X \to B$ such that $\operatorname{diam} \bar{g}(X) < \varepsilon$.

Since g and f are ξ-close, by our choice of ξ we may now conclude that f has an extension $\bar{f} : X \to Y$ such that $\bar{f}$ is λ-close to $\bar{g}$. Finally, by the choice of λ, $\operatorname{diam} \bar{f}(X) < \varepsilon$ and $\bar{f}(X) \subseteq O_\mu(B)$. □

6. Proof of Main Lemma

Proof. Let we are given a $\delta > 0$ and $\mathcal{V} \in \operatorname{cov}(X)$. We say, that cover $\mathcal{U} = \{U_\alpha : \alpha \in \mathfrak{A}\}$ and sequences $\{\mathcal{U}_k \in \operatorname{cov}(X) : k = 1 \dots n+1\}$, $\{x(k,\alpha) : \alpha \in \mathfrak{A}, k = 1 \dots n+1\}$ form *canonical system with respect to F and δ*, if the following conditions are satisfied (we use notation of Definitions 4.1, 4.2):

1. $\mathcal{U}_1 \equiv \mathcal{V}$.
2. $\mathcal{U}_{k+1}$ is canonical refinement of $\mathcal{U}_k$ with respect to δ and F.
3. $\mathcal{U}$ is canonical refinement of $\mathcal{U}_{n+1}$.
4. $\operatorname{St}_{\mathcal{U}} U_\alpha \subseteq U_{x(n+1,\alpha)} \in \mathcal{U}_{n+1}$ such that $x(n+1,\alpha) = x(U_{x(n+1,\alpha)})$.
5. $\operatorname{St}_{\mathcal{U}_{k+1}} U_{x(k+1,\alpha)} \subseteq U_{x(k,\alpha)} \in \mathcal{U}_k$ such that $x(k,\alpha) = x(U_{x(k,\alpha)})$.

Note, that canonical system exists for each $\mathcal{V} \in \operatorname{cov}(X)$. Note also, that some of $\{x(k,\alpha)\}$ may coincide.

Finally, observe, that since $\{F(x) : x \in X\}$ is $LC^{[L]}$ collection, we may assume without loss of generality that δ and ν which Lemmas 5.1, 5.2 provide us with for every $F(x)$ do not depend on x.

a. Fix $\mu > 0$. Construct sequence $\{\delta_k\}_{k=1}^{n+1}$ such that $\delta_{n+1} = \mu$ and for each k pair $(\delta_{k+1}/2, \delta_k)$ satisfies conditions of Lemma 5.2.**a** for any $F(x)$. Let $\delta = \delta_1/2$. In addition, we may assume that $\delta_k < \delta_{k+1}$ for every k.

Consider also a cover $\mathcal{V} = \{U_x : x \in X\}$, where $U_x = \{x' \in X : F(x) \subseteq O_\delta(F(x'))\}$.

Let $\mathcal{U} = \{U_\alpha : \alpha \in \mathfrak{A}\}$, $\{\mathcal{U}_k \in \operatorname{cov}(X) : k = 1 \dots n+1\}$, $\{x(k,\alpha) : \alpha \in \mathfrak{A}, k = 1 \dots n+1\}$ be canonical system for δ and $\mathcal{V}$.

For each $\alpha \in \mathfrak{A}$ pick $y_\alpha \in F(x(1,\alpha))$.

Finally, consider sets $\{F_k\}$ and $\{G_k^{\mathfrak{B}}\}$, constructed with respect to $\mathcal{U}$ (see Section 4).

Now we construct map g.

Using induction by $k = 1 \dots n+1$ construct a sequence of maps $\{g_k\}_{k=1}^{n+1}$ such that:

i. $g_k : F_k \to Y$.
ii. $g_{k+1}|_{F_k} = g_k$.

iii. g_k is δ_k-close to $F|_{F_k}$.
iv. For each $\mathfrak{B} \subseteq \mathfrak{A}$ such that $|\mathfrak{B}| = i \leq k$ there exists $\alpha = \alpha(\mathfrak{B}) \in \mathfrak{B}$ having property $g_k(G_i^{\mathfrak{B}}) \subseteq O_{\delta_i/2}(F(x(i, \alpha(\mathfrak{B}))))$.

For $k = 1$ define g_1 letting $g_1|_{G_1^{\{\alpha\}}} \equiv y(\alpha)$. Observe, that g_1 is defined correctly and continuous on F_1 (see properties G1–G4 on the page 5). By properties 1–4 of canonical system and by the choice of $\{y_\alpha\}$, g_1 satisfies requirements i–iv.

Assuming that g_k has been already constructed, let us construct g_{k+1}.

To accomplish this it is enough (see G3 on the page 5) to define g_{k+1} on each non-empty $G_{k+1}^{\mathfrak{B}}$ for each $\mathfrak{B} \subseteq \mathfrak{A}$ such that $|\mathfrak{B}| = k + 1$.

Fix $\mathfrak{B}$ such that $|\mathfrak{B}| = k+1$. Consider $Z = \overline{G_{k+1}^{\mathfrak{B}}}$. Let $Z' = Z \bigcap F_k$. Obviously, Z' is closed and non-empty subset of Z. The idea is to define g_{k+1} on $G_{k+1}^{\mathfrak{B}}$ extending g_k from Z' over Z.

For each $\mathfrak{B}' \subsetneqq \mathfrak{B}$ consider $\alpha(\mathfrak{B}')$ which exists by propery iv for map g_k. Consider $U_{x(k+1,\alpha(\mathfrak{B}'))}$.

By properties 4 and 5 of canonical system we have:

$$U_{\alpha(\mathfrak{B}')} \subseteq U_{x(k+1,\alpha(\mathfrak{B}'))}$$

(∗) and

$$\mathrm{St}_{\mathfrak{U}_{k+1}} U_{x(k+1,\alpha(\mathfrak{B}'))} \subseteq U_{x(k,\alpha(\mathfrak{B}'))} \subseteq U_{x(|\mathfrak{B}'|,\alpha(\mathfrak{B}'))}$$

Since $\bigcap\{U_{x(k+1,\alpha(\mathfrak{B}'))} : \mathfrak{B}' \subsetneqq \mathfrak{B}\} \supseteq G_{k+1}^{\mathfrak{B}} \neq \varnothing$ from (∗) we can conclude that there exists $\mathfrak{B}'' \subsetneqq \mathfrak{B}$ with the following property:

(∗∗)

$$U_{x(k+1,\alpha(\mathfrak{B}''))} \subseteq \bigcap\{U_{x(k+1,\alpha)} : \alpha \in \mathfrak{B}\} \subseteq \bigcap\{U_{x(|\mathfrak{B}'|,\alpha(\mathfrak{B}'))} : \mathfrak{B}' \subsetneqq \mathfrak{B}\}$$

Define $\alpha(\mathfrak{B}) = \alpha(\mathfrak{B}'')$.

Property (∗∗) coupled with property 2 of Definition 4.2 implies that for any $\mathfrak{B}' \subsetneqq \mathfrak{B}$ we have $F(x(k+1, \alpha(\mathfrak{B}'))) \subseteq O_\delta(F(x(k+1, \alpha(\mathfrak{B}))))$. Last inclusion and property iv of f_k (as well as our choice of the sequence $\{\delta_l\}$) yields the following chain of inclusions for each $\mathfrak{B}' \subsetneqq \mathfrak{B}$, $|\mathfrak{B}'| = i \leq k$:

$g_k(G_{|\mathfrak{B}'|}^{\mathfrak{B}'}) \subseteq O_{\delta_i/2+\delta}(F(x(k+1, \alpha(\mathfrak{B})))) \subseteq O_{\delta_k/2+\delta}(F(x(k+1, \alpha(\mathfrak{B})))) \subseteq O_{\delta_k}(F(x(k+1, \alpha(\mathfrak{B}))))$.

Hence (see property G4 on the page 5) $g_k(Z') \subseteq O_{\delta_k}(F(x(k+1, \alpha(\mathfrak{B}))))$.

The last fact and our choice of sequence $\{\delta_l\}$ allow us to extend g_k to g_{k+1} over Z such that

(∗∗∗) $$g_{k+1}(Z) \subseteq O_{\delta_{k+1}/2}(F(x(k+1, \alpha(\mathfrak{B}))))$$

Observe, that g_{k+1} is correctly defined and continuous on F_{k+1} by the properties G2 and G4 on the page 5. Let us check that g_{k+1} satisfies conditions i–iv.

Indeed, conditions i and ii are met by construction.

Further, since $G^{\mathfrak{B}}_{k+1} \subseteq Z$, condition iv follows from $(***)$. Finally, since $G^{\mathfrak{B}}_{k+1} \subseteq U_{x(k+1,\alpha(\mathfrak{B}))}$, from property 2 of Definition 4.2 applied to $\mathcal{U}_{k+1}$ we have $\operatorname{dist}(F|_{G^{\mathfrak{B}}_{k+1}}, g_{k+1}|_{G^{\mathfrak{B}}_{k+1}}) < \delta_{k+1}/2 + \delta < \delta_{k+1} < \mu$, which shows that property iii is also met.

Since $F_{n+1} = X$ (see property F3 on the page 5) we complete the proof letting $g \equiv g_{n+1}$.

b. Fix $\mu, \varepsilon > 0$. Construct sequences $\{\delta_k\}_{k=1}^{n+1}$, $\{\nu_k\}_{k=1}^{n+1}$ such that $\delta_{n+1} = \varepsilon$, $\nu_{n+1} = \mu$ and for each k we have $\delta_k = \delta(\delta_{k+1}/6)$, $\nu_k = \nu(\delta_{k+1}/6, \nu_{k+1}/2)$ in the sense of Lemma 5.2.**b** applied to $F(x)$ (for any $x \in X$).

Let $\delta = \delta_1/12$ and $\nu = \nu_1/2$. In addition, we may assume that $\delta_k < \delta_{k+1}/2$ and $\nu_k < \nu_{k+1}$ for every k.

Suppose that we are given a map $f : X \to Y$ such that f is δ-close to F.

Consider a cover $\mathcal{V} = \{V_x : x \in X\}$, where $V_x = \{x' \in X : F(x) \subseteq O_\nu(F(x'))\} \bigcap f^{-1}(O_\delta f(x))$.

Let $\mathcal{U} = \{U_\alpha : \alpha \in \mathfrak{A}\}$, $\{\mathcal{U}_k \in \operatorname{cov}(X) : k = 1 \ldots n+1\}$, $\{x(k,\alpha) : \alpha \in \mathfrak{A}, k = 1 \ldots n+1\}$ be canonical system for ν and $\mathcal{V}$.

Since $\operatorname{dist}(f, F) < \delta$, for each $\alpha \in \mathfrak{A}$ we can pick $y_\alpha \in F(x(1,\alpha))$ such that $\operatorname{dist}(y_\alpha, f(x(1,\alpha))) < \delta$.

Finally, consider sets $\{F_k\}$ and $\{G^{\mathfrak{B}}_k\}$, constructed with respect to $\mathcal{U}$ (see Section 4).

Now we construct map g.

As in part **a**, using induction by $k = 1 \ldots n+1$ construct a sequence of maps $\{g_k\}_{k=1}^{n+1}$ such that:

i. $g_k : F_k \to Y$.
ii. $g_{k+1}|_{F_k} = f_k$.
iii. g_k is ν_k-close to $F|_{F_k}$.
iv. For each $\mathfrak{B} \subseteq \mathfrak{A}$ such that $|\mathfrak{B}| = i \le k$ there exists $\alpha = \alpha(\mathfrak{B}) \in \mathfrak{B}$ having property $g_k(G^{\mathfrak{B}}_i) \subseteq O_{\nu_i/2}(F(x(i,\alpha(\mathfrak{B}))))$.
v. For each α, $g_k(F_k \bigcap U_\alpha) \subseteq O_{\delta_k/3}(y_\alpha)$

For $k = 1$ define g_1 letting $g_1|_{G_1^{\{\alpha\}}} \equiv y(\alpha)$. Observe, that g_1 is defined correctly and continuous on F_1 (see properties G1–G4 on the page 5). By properties 1–4 of canonical system and by the choice of $\{y_\alpha\}$, g_1 satisfies requirements i–v.

Assuming that g_k has been already constructed, let us construct g_{k+1}.

As before in the proof of **a**, to accomplish this it is enough (see G3 on the page 5) to define g_{k+1} on each non-empty $G^{\mathfrak{B}}_{k+1}$ for each $\mathfrak{B} \subseteq \mathfrak{A}$ such that $|\mathfrak{B}| = k+1$.

Fix $\mathfrak{B}$ such that $|\mathfrak{B}| = k+1$. Consider $Z = \overline{G^{\mathfrak{B}}_{k+1}}$. Let $Z' = Z \bigcap F_k$. Obviously, Z' is closed and non-empty subset of Z. Again, the idea is to define g_{k+1} on $G^{\mathfrak{B}}_{k+1}$ extending g_k from Z' over Z.

Using the same arguments as in proof of **a**, one can show, that

$$g_k(Z') \subseteq O_{\nu_k}(F(x(k+1, \alpha(\mathfrak{B})))) \qquad (')$$

Let us show, that

$$\operatorname{diam} g_k(Z') < \delta_k \qquad ('')$$

Indeed, since $\bigcap\limits_{\alpha \in \mathfrak{B}} U_\alpha \supseteq G^{\mathfrak{B}}_{k+1} \neq \varnothing$, we have $\bigcap\limits_{\alpha \in \mathfrak{B}} U_{x(1,\alpha)} \neq \varnothing$. Therefore $\operatorname{dist}(f(x(1,\alpha)), f(x(1,\beta))) < 2\delta$ for any $\alpha, \beta \in \mathfrak{B}$. Further, by construction we have $\operatorname{dist}(y_\alpha, f(x(1,\alpha))) < \delta$ for any $\alpha \in \mathfrak{B}$. These inequalities coupled with property v and the fact that $Z' \subseteq \bigcup\limits_{\alpha \in \mathfrak{B}} U_\alpha$ yield $\operatorname{diam} g_k(Z') < 2\delta + 2\delta + 2(\delta_k/3) = 4\delta + 2(\delta_k/3) < \delta_k$. Property ($''$) is checked.

From properties ($'$) and ($''$) we can conclude according to our choice of sequences $\{\delta_i\}$ and $\{\nu_i\}$ that g_k can be extended over Z to $g_{k+1} : Z \to Y$ such that

$$\operatorname{diam} g_{k+1}(Z) < \delta_{k+1}/6 \qquad (''')$$

Using the same arguments as in proof of **a** one can show that g_{k+1} is continuous map satisfying properties i–iv.

Let us check that property v is also met.

Since $Z' \bigcap U_\alpha \neq \varnothing$, for each $\alpha \in \mathfrak{B}$, from property ($'''$) and property v applied to map g_k we have:

$$g_{k+1}(F_{k+1} \bigcap U_\alpha) \subseteq O_{\delta_{k+1}/6 + \delta_k/3}(y_\alpha) \subseteq O_{\delta_{k+1}/3}(y_\alpha)$$

and condition v is checked.

Finally, let $g \equiv g_{n+1}$.

Obviously, g is μ-close to F. Check, that g is ε-close to f.

Property v of $g_{n+1} = g$ implies that $g_{n+1}(U_\alpha) \subseteq O_{\delta_{n+1}/3}(y_\alpha) = O_{\varepsilon/3}(y_\alpha)$. Since $U_\alpha \subseteq U_{x(1,\alpha)} \subseteq f^{-1}(O_\delta f(x(1,\alpha)))$, we have $f(U_\alpha) \subseteq O_\delta f(x(1,\alpha))$.

These inclusions coupled with inequality $\operatorname{dist}(f(x(1,\alpha)), y_\alpha) < \delta$ imply that $\operatorname{dist}(g|_{U_\alpha}, f|_{U_\alpha}) < 2\delta + \varepsilon/3 < \varepsilon$ for each α and consequently g is ε-close to f. □

The author is grateful to A. C. Chigogidze for attension to this work and useful discussions.

References

[1] E. Michael, *Continuous selections II*, Ann. Math. **63** (1956), 562-580.
[2] A. Chigogidze, *Infinite dimensional topology and shape theory*, to appear in: "Handbook of Geometric Topology" edited by R. Daverman and R. B. Sher), North Holland, Amsterdam, 1999.
[3] R. Engelking, *General Topology*, PWN, W rszawa, 1977.

Department of Mathematics and Statistics, University of Saskatchewan, McLean Hall, 106 Wiggins Road, Saskatoon, SK, S7N 5E6, Canada
E-mail address: karasev@math.usask.ca